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Minimizing ordered weighted averaging of rational functions with applications to continuous location

Víctor Blanco [a], Safae El Haj Ben Ali [b], Justo Puerto [b,*]

[a] Departamento de Métodos Cuantitativos para la Economía y la Empresa, Universidad de Granada, 18011, Spain
[b] Departamento de Estadística e Investigación Operativa, Universidad de Sevilla, 41012, Spain

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ABSTRACT

This paper considers the problem of minimizing the ordered weighted average (or ordered median) function of finitely many rational functions over compact semi-algebraic sets. Ordered weighted averages of rational functions are, in general, neither rational functions nor the supremum of rational functions so current results available for the minimization of rational functions cannot be applied to handle these problems. We prove that the problem can be transformed into a new problem embedded in a higher dimensional space where it admits a convenient polynomial optimization representation. This reformulation allows a hierarchy of SDP relaxations that approximates, up to any degree of accuracy, the optimal value of those problems. We apply this general framework to a broad family of continuous location problems showing that some difficult problems (convex and non-convex) that up to date could only be solved on the plane and with Euclidean distance can be reasonably solved with different ℓ_p-norms in finite dimensional spaces. We illustrate this methodology with some extensive computational results on constrained and unconstrained location problems.

1. Introduction

Ordered weighted averaging (OWA) or ordered median function (OMF) operators provide a parameterized class of mean type aggregation operators (see [28,50] and the references therein for further details). Many notable mean operators, such as the maximum, arithmetic average, median, k-centrum, range and minimum, are members of this class. They have been widely used in location theory and artificial intelligence because of their ability to represent flexible models of modern logistics and linguistically expressed aggregation instructions in artificial intelligence [28,45–50]. Ordered weighted averages (or ordered median) of rational functions are, in general, neither rational functions nor the supremum of rational functions so current results available for the minimization of rational functions are not applicable. In spite of its intrinsic interest, one can only find in the literature different methods for solving particular instances, see e.g. [6,7,17,28–34,36,38], nevertheless as far as we know, a common approach for solving this family of problems is not available yet. The first goal of this paper is to develop a unified tool for solving this class of optimization problems. In this line, we prove that the general problem can be transformed into a new problem embedded in a higher dimensional space where it admits a convenient polynomial optimization representation that allows to arbitrarily approximate or to solve it as a minimization problem over an adequate closed semi-algebraic set.

Regarding the applications, it is commonly agreed that ordered median location problems are among the most important applications of OWA operators. Continuous location problems appear very often in economic models of distribution or logistics, in statistics when one tries to find an estimator from a data set or in pure optimization problems where one looks for the optimizer of a certain function. For a comprehensive overview of the use of these operators in optimization the reader is referred to [5] or [28]. Despite the fact that many continuous location problems rely heavily on a common framework, specific solution approaches have been developed for each of the typical objective functions in location theory (see for instance [5]). To overcome this inflexibility and to work towards a unified approach to location theory the so-called ordered median problem (OMP) was developed (see [28] and references therein). Ordered median problems represent as special cases nearly all classical objective functions in location theory. More precisely, the 1-facility ordered median problem can be formulated as follows: a vector of weights $(\lambda_1, \ldots, \lambda_n)$ is given. The problem is to find a location for a facility that minimizes the weighted sum of distances where the distance to the closest point to the facility is multiplied by the weight λ_n, the distance to the second closest, by λ_{n-1}, and so on. The distance to the farthest point is multiplied by λ_1. Many location problems can be formulated as the ordered 1-median problem by selecting appropriate weights. For example, the vector for which all $\lambda_i = 1$ is the 1-median problem, the problem

* Corresponding author.
E-mail addresses: vblanco@ugr.es (V. Blanco), anasafae@gmail.com (S.E. Ben Ali), puerto@us.es (J. Puerto).

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where $\lambda_1=1$ and all others are equal to zero is the 1-center problem, the problem where $\lambda_1=\cdots=\lambda_k=1$ and all others are equal to zero is the k-centrum. Minimizing the range of distances is achieved by $\lambda_1=1$, $\lambda_n=-1$ and all others are zero. Despite its full generality, the main drawback of this framework is the difficulty of solving the problems with a unified tool. There are nowadays some successful approaches available whenever the framework space is either discrete (see [3,25,33]) or a network (see [14,15] or [27]). Nevertheless, the continuous case has been, so far, only partially covered and there have been some attempts to overcome this drawback, at least in the plane and with Euclidean norm. For instance, in Drezner [4] and Drezner and Nickel [6,7] the authors present two different approaches. The first one uses a continuous branch and bound method based on triangulations (BTST) and the second one on a D-C decomposition for the objective function that allow solving the problems on the plane. More recently, Espejo et al. [11] and Rodriguez-Chia et al. [38] also address the particular case of the planar convex ordered and k-centrum problems with Euclidean distances by using geometric arguments to develop better algorithms for those problems.

Our aim is to design a common approach to solve the above mentioned family of location problems, for different distances in finite dimensional spaces. This is essentially the second goal of this paper. In our way, we have addressed the more general problem that consists of the minimization of the OWA operator of a finite number of rational functions over basic, closed semi-algebraic sets. Of course, we know that this problem in its full generality is *NP-hard* since it includes general instances of concave minimization (see e.g. [35]). Therefore, we cannot expect to find polynomial time algorithms for this class of problems. Rather, we will apply a methodology first proposed by Lasserre [19] that provides a hierarchy of semidefinite programming problems (in short, SDP) that converges to the optimal solution of the original problem, with the property that each auxiliary problem in the sequence can be solved in polynomial time. (See e.g. [42] for some classical complexity results on semidefinite programming.)

The paper is organized in six sections. The first one is our introduction. In the second section and for the sake of completeness, we recall some general results on the Theory of Moments that will be useful in the rest of the paper. Section 3 introduces the problem $\mathbf{OMRP}_\lambda$ which consists of minimizing the *ordered median function* of finitely many rational functions over a compact basic semi-algebraic set. In the spirit of the moment approach developed in Lasserre [19,21] for polynomial optimization and later adapted by Jibetean and De Klerk [13], we define a hierarchy of semidefinite programming relaxations. Each SDP relaxation is a semidefinite program which, up to arbitrary (but fixed) precision, can be solved in polynomial time and the monotone sequence of optimal values associated with the hierarchy converges to the optimal value of $\mathbf{OMRP}_\lambda$. Under some quite general conditions, the convergence is finite and we can even detect whether a certain relaxation in the hierarchy is exact (i.e. provides the optimal value), and to extract optimal solutions (theoretical bounds on the relaxation order for the exact results can be found in [40,41]). Section 4 considers a general family of location problems that is built from the problem $\mathbf{OMRP}_\lambda$ but which does not actually fits under the same formulation because the objective functions are not quotients of polynomials. Nevertheless, we prove that under a certain reformulation one can define another hierarchy of relaxed problems that fulfills similar convergence properties. This approach is applicable to location problems with ℓ_p-norms (for $p\in\mathbb{Q}$) in finite dimensional spaces. We exploit the special structure of these problems to find a block diagonal reformulation that reduces the sizes of the SDP relaxations and allows to solve larger instances. Our computational tests are presented in Section 5 and they are organized to be comparable with previous results presented in the literature. We analyze five families of problems, namely, Weber, center, k-centrum, trimmed-mean and range in $\mathbb{R}^2$ and $\mathbb{R}^3$ with norms ℓ_2 and ℓ_3. There we show that convergence is rather fast and high accuracy is achieved in all cases, in low order relaxations. (We prove that for location problems with Euclidean distances that relaxation order is $r=2$.) In addition, we include some computational results on non-convex constrained location problems in $\mathbb{R}^3$ and with the ℓ_3-norm to show the powerfulness of our approach to also handle non-convex problems. The paper ends with some conclusions and an outlook for further research.

2. Preliminaries

In this section we recall the main definitions and results on the *Problem of Moments* that will be useful for the development through this paper. We use standard notation in the field (see e.g. [23]).

We denote by $\mathbb{R}[x]$ the ring of real polynomials in the variables $x=(x_1,\ldots,x_d)$, for $d\in\mathbb{N}$ $(d\geq 1)$, and by $\mathbb{R}[x]_r\subset\mathbb{R}[x]$ the space of polynomials of degree at most $r\in\mathbb{N}$ (here $\mathbb{N}$ denotes the set of non-negative integers). We also denote by $\mathcal{B}=\{x^\alpha:\alpha\in\mathbb{N}^d\}$ a canonical basis of monomials for $\mathbb{R}[x]$, where $x^\alpha=x_1^{\alpha_1}\cdots x_d^{\alpha_d}$, for any $\alpha\in\mathbb{N}^d$. Note that $\mathcal{B}_r=\{x^\alpha\in\mathcal{B}:\sum_{i=1}^d\alpha_i\leq r\}$ is a basis for $\mathbb{R}[x]_r$.

For any sequence indexed in the canonical monomial basis $\mathcal{B}$, $\mathbf{y}=(y_\alpha)_{\alpha\in\mathbb{N}^d}\subset\mathbb{R}$, let $L_\mathbf{y}:\mathbb{R}[x]\to\mathbb{R}$ be the linear functional defined, for any $f=\sum_{\alpha\in\mathbb{N}^d}f_\alpha x^\alpha\in\mathbb{R}[x]$, as $L_\mathbf{y}(f):=\sum_{\alpha\in\mathbb{N}^d}f_\alpha y_\alpha$.

The *moment* matrix $M_r(\mathbf{y})$ of order r associated with $\mathbf{y}$ has its rows and columns indexed by (x^α) and $M_r(\mathbf{y})(\alpha,\beta):=L_\mathbf{y}(x^{\alpha+\beta})=y_{\alpha+\beta}$, for $|\alpha|,|\beta|\leq r$ (here $|a|$ stands for the sum of the coordinates of $a\in\mathbb{N}^d$). Note that the moment matrix of order r has dimension $\binom{d+r}{d}\times\binom{d+r}{d}$ and that there are $\binom{d+2r}{d}\mathbf{y}_\alpha$ variables.

For $g\in\mathbb{R}[x](=\sum_{\gamma\in\mathbb{N}^d}g_\gamma x^\gamma)$, the *localizing* matrix $M_r(g\mathbf{y})$ of order r associated with $\mathbf{y}$ and g has its rows and columns indexed by (x^α) and $M_r(g\mathbf{y})(\alpha,\beta):=L_\mathbf{y}(x^{\alpha+\beta}g(x))=\sum_\gamma g_\gamma y_{\gamma+\alpha+\beta}$, for $|\alpha|,|\beta|\leq r$.

Definition 1. Let $\mathbf{y}=(y_\alpha)\subset\mathbb{R}$ be a sequence indexed in the canonical monomial basis $\mathcal{B}$. We say that $\mathbf{y}$ has a *representing* measure supported on a set $\mathbf{K}\subseteq\mathbb{R}^d$ if there is some finite Borel measure μ on $\mathbf{K}$ such that

$$y_\alpha=\int_\mathbf{K}x^\alpha\,d\mu(x)\quad\text{for all }\alpha\in\mathbb{R}^d.$$

The tools used to ensure that a sequence of moments has a representing Borel measure are based on finding certificates of positivity for real polynomials. The first representation result, known as Schmudgen's Positivstellensatz, was an important breakthrough in the field [39]. An improvement on that result, under a relatively weak assumption called Archimedean condition is due to Putinar [37]. The main assumption to be imposed when one wants to assure the convergence of the SDP relaxations for solving polynomial optimization problems (see for instance [22,23]) is a consequence of Putinar's results [37] and it is stated as follows.

Archimedean property. Let $\{g_1,\ldots,g_l\}\subset\mathbb{R}[x]$ and $\mathbf{K}:=\{x\in\mathbb{R}^d: g_j(x)\geq 0, j=1,\ldots,\ell\}$ a basic closed semi-algebraic set. Then, $\mathbf{K}$ satisfies Archimedean property if there exists $u\in\mathbb{R}[x]$ such that:

1. $\{x:u(x)\geq 0\}\subset\mathbb{R}^d$ is compact, and
2. $u=\sigma_0+\sum_{j=1}^\ell\sigma_j\,g_j$, for some $\sigma_1,\ldots,\sigma_l\in\Sigma[x]$. (This expression is usually called *Putinar's representation* of u over $\mathbf{K}$).

Being $\Sigma[x]\subset\mathbb{R}[x]$ the subset of polynomials that are sums of squares.

Note that Archimedean property is equivalent to impose that the quadratic polynomial $u(x)=M-\sum_{i=1}^{d}x_i^2$ has Putinar's representation over $\mathbf{K}$ for some $M>0$.

We observe that Archimedean property implies compactness of $\mathbf{K}$. It is easy to see that Archimedean property holds if either $\{x: g_j(x)\geq 0\}$ is compact for some j, or all g_j are affine and $\mathbf{K}$ is compact. Furthermore, Archimedean property is not restrictive at all, since any semi-algebraic set $\mathbf{K}\subseteq\mathbb{R}^d$ for which is known that $\sum_{i=1}^{d}x_i^2\leq M$ holds for some $M>0$ and for all $x\in\mathbf{K}$, admits a new representation $\mathbf{K}'=\mathbf{K}\cup\{x\in\mathbb{R}^d: g_{l+1}(x):=M-\sum_{i=1}^{d}x_i^2\geq 0\}$ that verifies Archimedean property (see Section 2 in [23]).

The importance of Archimedean property stems from the following result that links such a condition with the semidefiniteness of the moment and localizing matrices.

Theorem 2 (*Putinar [37]*). *Let $\{g_1,\ldots,g_l\}\subset\mathbb{R}[x]$ and $\mathbf{K}:=\{x\in\mathbb{R}^d: g_j(x)\geq 0,\ j=1,\ldots,\ell\}$ satisfying Archimedean property. Then:*

1. *Any $f\in\mathbb{R}[x]$ which is strictly positive on $\mathbf{K}$ has Putinar's representation over $\mathbf{K}$.*
2. *$\mathbf{y}=(y_\alpha)$ has a representing measure on $\mathbf{K}$ if and only if $M_r(\mathbf{y})\succcurlyeq 0$, and $M_r(g_j\mathbf{y})\succcurlyeq 0$, for all $j=1,\ldots,l$ and $r\in\mathbb{N}$.*

(*Here, the symbol $\succcurlyeq 0$ stands for positive semidefinite matrix.*)

The following result that appears in [18] will be crucial for the development in the next sections.

Lemma 3 (*Laraki and Lasserre [18, Lemma 2.3]*). *Let $\mathbf{K}\subset\mathbb{R}^d$ be compact and let p, q be continuous with $q>0$ on $\mathbf{K}$. Let $\mathcal{M}(\mathbf{K})$ be the set of finite Borel measures on $\mathbf{K}$ and let $\mathcal{P}(\mathbf{K})\subset\mathcal{M}(\mathbf{K})$ be its subset of probability measures on $\mathbf{K}$. Then*

$$\min_{\mu\in\mathcal{P}(\mathbf{K})}\frac{\int_{\mathbf{K}}p\,d\mu}{\int_{\mathbf{K}}q\,d\mu}=\min_{\varphi\in\mathcal{M}(\mathbf{K})}\left\{\int_{\mathbf{K}}p\,d\varphi:\int_{\mathbf{K}}q\,d\varphi=1\right\}$$
$$=\min_{\mu\in\mathcal{P}(\mathbf{K})}\int_{\mathbf{K}}\frac{p}{q}\,d\mu=\min_{x\in\mathbf{K}}\frac{p(x)}{q(x)}.$$

3. Minimizing the ordered weighted average of finitely many rational functions

Let $\mathbf{K}\subset\mathbb{R}^d$ be a basic, closed semi-algebraic set defined as

$$\mathbf{K}:=\{x\in\mathbb{R}^d: g_j(x)\geq 0,\ j=1,\ldots,\ell\} \tag{1}$$

for $g_1,\ldots,g_\ell\in\mathbb{R}[x]$. We assume that $\mathbf{K}$ satisfies Archimedean property.

Let us introduce the ordered median function $\mathrm{OM}(x)=\sum_{k=1}^{m}\lambda_k(x)f_{(k)}(x)$, for some rational functions $(f_j)\subset\mathbb{R}[x]$, being $f_k=p_k/q_k$ rational functions with $p_k,q_k\in\mathbb{R}[x]$, $q_k>0$ on $\mathbf{K}$, for every $k=1,\ldots,m$. In addition, let $\lambda_k(x)\in\mathbb{R}[x]$ be generic polynomials and $f_{(k)}(x)\in\{f_1(x),\ldots,f_m(x)\}$ such that $f_{(1)}(x)\geq f_{(2)}(x)\ \geq\cdots\geq f_{(m)}(x)$ for $x\in\mathbb{R}^d$.

Consider the following problem:

$$\rho_\lambda:=\min_{x\in\mathbf{K}}\mathrm{OM}(x). \tag{$\mathbf{OMRP}^0_\lambda$}$$

Associated with the above minimization problem we introduce an equivalent formulation that will be useful to apply the moment tools to solve the ordered median problem. For each $i=1,\ldots,m$, $j=1,\ldots,m$ consider the following family of decision variables for each $x\in\mathbf{K}$:

$$w_{ij}=\begin{cases}1 & \text{if } f_i(x)=f_{(j)}(x),\\ 0 & \text{otherwise}.\end{cases}$$

Now, we consider the problem:

$$\overline{\rho}_\lambda=\min_{x,w}\sum_{j=1}^{m}\lambda_j(x)\sum_{i=1}^{m}f_i(x)w_{ij} \tag{$\mathbf{OMRP}_\lambda$}$$

$$\text{s.t.}\quad\sum_{j=1}^{m}w_{ij}=1\quad\text{for } i=1,\ldots,m, \tag{2}$$

$$\sum_{i=1}^{m}w_{ij}=1\quad\text{for } j=1,\ldots,m,$$
$$w_{ij}^2-w_{ij}=0\quad\text{for } i,j=1,\ldots,m,$$
$$h_j(x,w):=\sum_{i=1}^{m}w_{ij}f_i(x)-\sum_{i=1}^{m}w_{ij+1}f_i(x)\geq 0,\quad j=1,\ldots,m-1, \tag{3}$$

$$h'_j(w):=1-\sum_{i=1}^{m}w_{ij}^2\geq 0,\quad j=1,\ldots,m \tag{4}$$

$$w_{ij}\in\mathbb{R}\quad\text{for } i,j=1,\ldots,m,\ x\in\mathbf{K}. \tag{5}$$

The first set of constraints ensures that for each x, $f_i(x)$ is sorted in a unique position. The second set ensures that the jth position is only assigned to one rational function. The next constraints are added to assure that $w_{ij}\in\{0,1\}$. The fourth one states that $f_{(1)}(x)\geq\cdots\geq f_{(m)}(x)$. The last set of constraints ensures that Archimedean property holds for the new feasible region. (Note that this last set of constraints are redundant but it is convenient to add them for a better description of the feasible set.)

These two problems ($\mathbf{OMRP}^0_\lambda$) and ($\mathbf{OMRP}_\lambda$) satisfy the following relationship.

Theorem 4. *Let $x\in\mathbf{K}$ be a feasible solution of ($\mathbf{OMRP}^0_\lambda$) then there exists $w\in\{0,1\}^{m\times m}$, fulfilling that (x,w) is a feasible solution of ($\mathbf{OMRP}_\lambda$) and such that both solutions have equal objective values. Conversely, if $(x,w)\in\mathbf{K}\times\mathbb{R}^{m\times m}$ is a feasible solution for ($\mathbf{OMRP}_\lambda$) then x is a feasible solution of ($\mathbf{OMRP}^0_\lambda$) and both solutions have the same objective value. In particular $\rho_\lambda=\overline{\rho}_\lambda$.*

Proof. Let $\overline{x}$ be a feasible solution of ($\mathbf{OMRP}^0_\lambda$). Then, it clearly satisfies that $\overline{x}\in\mathbf{K}$. In addition, let σ be the permutation of $\{1,\ldots,m\}$ such that $f_{\sigma(1)}(\overline{x})\geq f_{\sigma(2)}(\overline{x})\geq\cdots\geq f_{\sigma(m)}(\overline{x})$. Take,

$$\overline{w}_{ij}=\begin{cases}1 & \text{if } i=\sigma(j),\\ 0 & \text{otherwise}.\end{cases}$$

Clearly, $(\overline{x},\overline{w})$ satisfy the constraints in (2)–(5). Indeed, for any i, $\sum_{j=1}^{m}\overline{w}_{ij}=\overline{w}_{i\sigma^{-1}(i)}=1$. Analogously, for any j, $\sum_{i=1}^{m}\overline{w}_{ij}=\overline{w}_{\sigma(j),j}=1$ (being then (4) also satisfied). By its own definition, $\overline{w}$ only takes 0,1 values and thus, $\overline{w}_{ij}^2-\overline{w}_{ij}=0$ for all i,j. Finally, to prove that $(\overline{x},\overline{w})$ satisfies (3), we observe, w.l.o.g., that for any j there exist i^* and $\hat{i}$ such that $\sigma(j)=i^*$ and $\sigma(j+1)=\hat{i}$. Hence:

$$\sum_{i=1}^{m}\overline{w}_{ij}f_i(\overline{x})=\overline{w}_{i^*j}f_{\sigma(j)}(\overline{x})\geq\overline{w}_{\hat{i}j+1}f_{\sigma(j+1)}(\overline{x})=\sum_{i=1}^{m}\overline{w}_{ij+1}f_i(\overline{x}).$$

Moreover,

$$\mathrm{OM}_\lambda(\overline{x})=\sum_{j=1}^{m}\lambda_j(\overline{x})\sum_{i=1}^{m}f_i(\overline{x})\overline{w}_{ij}.$$

Conversely, if $(\overline{x},\overline{w})$ is a feasible solution of ($\mathbf{OMRP}_\lambda$) then, clearly $\overline{x}$ is feasible of ($\mathbf{OMRP}^0_\lambda$) and by the above, $\mathrm{OM}_\lambda(\overline{x})=\sum_{j=1}^{m}\lambda_j(\overline{x})\sum_{i=1}^{m}f_i(\overline{x})\overline{w}_{ij}$. $\square$

Next, we observe that since $f_i=p_i/q_i$ for each $i=1,\ldots,m$, both the objective function and the constraint (3) are rational functions. Moreover, the constraint $\sum_{i=1}^{m}w_{ij}f_i(x)\geq\sum_{j=1}^{m}w_{ij+1}f_i(x)$

can be written as a polynomial constraint in the form

$$\sum_{i=1}^{m} w_{ij} p_i(x) \prod_{k\neq i}^{m} q_k(x) \geq \sum_{i=1}^{m} w_{ij+1}\, p_i(x) \prod_{k \neq i}^{m} q_k(x), \quad j=1,\ldots,m.$$

Let us denote by $\overline{\mathbf{K}}$ the basic closed semi-algebraic set that defines the feasible region of ($\mathbf{OMRP}_\lambda$), namely:

$$\overline{\mathbf{K}} := \Bigg\{ (x,w) \in \mathbb{R}^{d+m^2} : \sum_{j=1}^{m} w_{ij}=1,\ \sum_{i=1}^{m} w_{ij}=1,\ w_{ij}^2 - w_{ij} = 0, \\ \forall\ i,j=1,\ldots,m, \quad h_j(x,w)\geq 0,\ j=1,\ldots,m-1, \\ h'_j(w)\geq 0,\ j=1,\ldots,m, \quad x\in \mathbf{K} \Bigg\}. \tag{6}$$

Note that $\overline{\mathbf{K}}$ has not the shape of a standard semi-algebraic set, as defined at the beginning of the section, since it is defined by equality constraints. However, it is clear that each equality constraint in $\overline{\mathbf{K}}$ can be equivalently written as two inequality constraints, being $\overline{\mathbf{K}}$ written as the semi-algebraic set in (1).

Proposition 5. *Let $\overline{\mathbf{K}} \subset \mathbb{R}^{d+m^2}$ be the closed basic semi-algebraic set defined in (6). If $\mathbf{K}$ satisfies Archimedean condition then $\overline{\mathbf{K}}$ also satisfies this condition. Moreover, consider the infinite dimensional optimization problem*

$$\mathcal{P}_\lambda : \quad \widehat{\rho}_\lambda = \min_{x,w} \left\{ \int_{\overline{\mathbf{K}}} p_\lambda \, d\mu : \int_{\overline{\mathbf{K}}} q_\lambda \, d\mu = 1, \mu \in M(\overline{\mathbf{K}}) \right\}, \tag{7}$$

being

$$p_\lambda(x,w) = \sum_{j=1}^{m} \lambda_j(x) \sum_{i=1}^{m} w_{ij}\, p_i(x) \prod_{k\neq i}^{m} q_k(x) \quad \text{and} \quad q_\lambda(x,w) = \prod_{k=1}^{m} q_k(x). \tag{8}$$

Then $\rho_\lambda = \widehat{\rho}_\lambda$.

Proof. Since $\mathbf{K}$ satisfies Archimedean property, the quadratic polynomial $x \mapsto u(x) := M - \|x\|_2^2$ can be written as $u(x) = \sigma_0(x) + \sum_{j=1}^{\ell} \sigma_j(x) g_j(x)$ for some s.o.s. polynomials $(\sigma_j) \subset \Sigma[x]$ and for some $M>0$. Next, consider the polynomial

$$(x,w) \mapsto r(x,w) = M + m - \|x\|_2^2 - \sum_{i=1}^{m}\sum_{j=1}^{m} w_{ij}^2.$$

Obviously, its level set $\{(x,w) \in \mathbb{R}^{d\times m^2} : r(x,z) \geq 0\} \subset \mathbb{R}^{n+m^2}$ is compact and moreover, r can be written in the form

$$r(x,w) = \sigma_0(x) + \sum_{j=1}^{\ell} \sigma_j(x) g_j(x) + 1 \times \sum_{j=1}^{m} \overbrace{\left(1 - \sum_{i=1}^{m} w_{ij}^2\right)}^{\overline{h}'_j(w) \text{ defining } \overline{\mathbf{K}}}$$

for appropriate s.o.s. polynomials $(\sigma'_j) \subset \Sigma[x,w]$ $(\sigma'_j = 1, \forall j)$. Therefore $\overline{\mathbf{K}}$ satisfies Archimedean property. In particular, this implies that $\overline{\mathbf{K}}$ is compact.

Now, we observe that the objective function of ($\mathbf{OMRP}_\lambda$) can be written as a quotient of polynomials in $\mathbb{R}[x,w]$. Indeed,

$$\sum_{j=1}^{m} \lambda_j(x) \sum_{i=1}^{m} f_i(x) w_{ij} = \frac{p_\lambda(x,w)}{q_\lambda(x,w)}. \tag{9}$$

Then, by Lemma 3 we can transform Problem ($\mathbf{OMRP}_\lambda$) in an infinite dimensional linear program on the space of Borel measures defined on $\overline{\mathbf{K}}$. It follows by applying Lemma 3 to the reformulation of ($\mathbf{OMRP}_\lambda$) with the objective function written in (9) using p_λ and q_λ in (8). □

The reader may note the generality of this class of problems. Depending on the choice of the polynomial weights $\lambda_1(x), \ldots, \lambda_m(x)$ we get different classes of problems. Among then, we emphasize the important instances given by:

1. $\lambda = (1,0,\ldots,0,0)$ which corresponds to minimize the maximum of a finite number of rational functions,
2. $\lambda = (1,\overset{(k)}{\ldots},1,0,\ldots,0)$ which corresponds to minimize the sum of the k-largest rational functions (k-centrum)
3. $\lambda = (0,\overset{(k_1)}{\ldots},0,1,\ldots,1,0,\overset{(k_2)}{\ldots},0)$ which models the minimization of the (k_1,k_2)-trimmed-mean of m rational functions,…
4. $\lambda = (1,\alpha,\ldots,\alpha)$ which corresponds to the α-centdian, i.e. minimizing the convex combination of the sum and the maximum of the set of rational functions.
5. $\lambda = (1,0,\ldots,0,-1)$ which corresponds to minimize the range of a set of rational functions.

Remark 6. Problem ($\mathbf{OMRP}_\lambda^0$) can be easily extended to deal with the minimization of the ordered median function of a finite number of several ordered median operators of rational functions. The reader may observe that this can be done by performing a similar transformation to the one in ($\mathbf{OMRP}_\lambda$) and thus lifting the original problem into a higher dimensional space.

3.1. Some remarkable special cases

The above general analysis extends the general theory of moments to the case of ordered weighted averages of rational functions. Notice that this approach goes beyond a trivial adaptation of that theory since ordered weighted averages of rational functions are not, in general, either rational functions or the supremum of rational functions so that current results cannot be applied to handle these problems. However, one can transform the problem into a new problem embedded in a higher dimensional space where it admits a representation that can be cast in the minimization of another rational function in a convenient closed semi-algebraic set. Needless to say that the number of variables increases with respect to the original one. This may become a problem in particular implementations due to the current state of nonlinear programming solvers.

Moreover, in some important particular cases that have been extensively considered in the field of Operations Research, the above approach can be further simplified as we will show in the following. One of these cases, the minimization of the maximum of finitely many rational functions has been already analyzed by Laraki and Lasserre [18]. We will show that such an approach is also a particular case of the analysis that we present in the following.

For the rest of this subsection we will restrict ourselves, for the sake of readability, to the case of scalar (real) lambda-weights. We will begin with the case of $\lambda = (1,\overset{(k)}{\ldots},1,0\ldots,0)$, for $1 \leq k \leq m$. Note that if $k=1$, we will recover the case analyzed in [18], the case $k=m$ is trivial since it reduces to minimize the overall sum but the remaining cases are not yet known.

We are interested in finding the minimum of the sum of the k-largest values $\{f_1(x),\ldots,f_m(x)\}$ for all $x \in \mathbf{K}$, being $\mathbf{K}$ the basic, closed semi-algebraic set defined in (1). In other words, for any $k \in \{1,\ldots,m-1\}$, we wish to solve the problem:

$$\varrho_k := \min_{x \in \mathbf{K}} S_k(x) := \sum_{j=1}^{k} f_{(j)}(x). \tag{10}$$

We observe that for a given $x \in \mathbf{K}$, we have

$$S_k(x) = \sum_{j=1}^{k} f_{(j)}(x) = \max\left\{ \sum_{j=1}^{m} \upsilon_j f_j(x) : \sum_{j=1}^{m} \upsilon_j = k, 0 \leq \upsilon_j \leq 1,\ \forall j \right\}.$$

Therefore, by the Strong Duality Theorem of linear programming:

$$S_k(x) = \min\left\{ kt + \sum_{j=1}^{m} r_j : t + r_j \geq f_j(x),\ r_j \geq 0,\ \forall j \right\}.$$

Finally, we consider the problem:

$$\begin{aligned} \hat{\varrho}_k \coloneqq \min \quad & kt + \sum_{j=1}^{m} r_j \\ \text{s.t.} \quad & t + r_j \geq f_j(x), \quad j = 1, \ldots, m, \\ & r_j \geq 0, \quad j = 1, \ldots, m, \\ & x \in \mathbf{K}. \end{aligned} \tag{kC}$$

Since we have assumed $\mathbf{K}$ to satisfy Archimedean condition, it follows that $\mathbf{K}$ is compact. Thus, for any $j = 1, \ldots, m$, there exist lower and upper bounds, LB_j and UB_j, for each f_j such that for any $x \in \mathbf{K}$,

$$LB_j \leq f_j(x) \leq UB_j, \quad j = 1, \ldots, m, \ \forall x \in \mathbf{K}.$$

Let us denote $LB = \min_{j=1..m} LB_j$ and $UB = \max_{j=1..m} UB_j$. Set

$$\overline{\mathbf{K}}_1 \coloneqq \{(x,t,r) \in \mathbb{R}^{d+1+m} : t + r_j \geq f_j(x),\ r_j \geq 0,\ r_j^2 \leq (UB_j - LB)^2,\ j = 1, \ldots, m,\\ t^2 \leq \max\{UB^2, LB^2\},\ x \in \mathbf{K}\}.$$

We get the following result.

Lemma 7. *If* $\mathbf{K} \subset \mathbb{R}^d$ *satisfies Archimedean property then* $\overline{\mathbf{K}}_1 \subset \mathbb{R}^{d+1+m}$ *satisfies Archimedean property. Moreover* $\varrho_k = \hat{\varrho}_k = \min \{kt + \sum_{j=1}^m r_j : (x,t,r) \in \overline{\mathbf{K}}_1\}$.

Proof. Consider an arbitrary k, $1 \leq k \leq m-1$ and an arbitrary (but fixed) $\overline{x} \in \mathbf{K}$. Without loss of generality, assume that $f_1(\overline{x}) \geq \cdots \geq f_m(\overline{x})$. We define the function

$$g(\overline{x}, t) \coloneqq \min\left\{ kt + \sum_{j=1}^{m} r_j : t + r_j \geq f_j(\overline{x}),\ r_j \geq 0,\ \forall\, j = 1, \ldots, m \right\}.$$

Clearly, $g(\overline{x}, t)$ is piecewise linear and convex as a function of t; and it attains its minimum at any point of the interval $I_k = (f_{k+1}(\overline{x}), f_k(\overline{x})]$. Indeed, observe that for any $t \in I_k$, the slope of g (i.e. its derivative with respect to t) is null since:

$$g(\overline{x}, t) = kt + \sum_{j=1}^{k} (f_j(\overline{x}) - t) = \sum_{j=1}^{k} f_j(\overline{x}) = S_k(\overline{x}).$$

From the above, we observe that

$$\begin{aligned} \varrho_k &= \min_{x \in \mathbf{K}} S_k(x) \\ &= \min_{\substack{x \in \mathbf{K} \\ t \in I_k}} g(x,t) \\ &= \min\left\{ kt + \sum_{j=1}^{m} r_j : t + r_j \geq f_j(x),\ r_j \geq 0,\ \forall\, j = 1, \ldots, m, x \in \mathbf{K},\ t \in I_k \right\} \\ &= \hat{\varrho}_k. \end{aligned}$$

It remains to prove that the constraints $r_j^2 \leq (UB_j - LB)^2$, $j = 1, \ldots, m$ and $t^2 \leq \max\{UB^2, LB^2\}$ are redundant for Problem (kC) and that the new feasible region $\overline{\mathbf{K}}_1$ satisfies Archimedean condition. First, we observe from the argument above that in order to obtain the minimum value of the function g, for any $k = 1, \ldots, m-1$ and any $x \in \mathbf{K}$, we only need to consider the range $t \in (f_{(m)}(x), f_{(1)}(x)]$. Hence, the overall range for t can be restricted to $LB \leq t \leq UB$, therefore any optimal solution of Problem (kC) satisfies $t^2 \leq \max\{UB^2, LB^2\}$. In addition, for any $x \in \mathbf{K}$, the constraints $r_j \geq f_j(x) - t$ define the range of the variable r_j and since we are minimizing r_j with positive coefficient, this variable will not be greater than the maximum of its lower limit. Hence,

$$0 \leq r_j \leq \max_{\substack{x \in \mathbf{K} \\ t \in [f_{(m)}(x), f_{(1)}(x)]}} f_j(x) - t \leq UB_j - LB, \quad \forall\, j = 1, \ldots, m.$$

Augmenting the constraints, $r_j^2 \leq (UB_j - LB)^2$, $\forall\, j = 1, \ldots, m$, and $t^2 \leq \max(UB^2, LB^2)$, in the definition of $\mathbf{K}$ does not change the value of $\hat{\varrho}_k$ and makes the feasible set $\overline{\mathbf{K}}_1$ to satisfy Archimedean condition using an argument similar to the one in the proof of Proposition 5. □

This approach extends further to the more general case of non-increasing monotone lambda-weights, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq \lambda_{m+1} \coloneqq 0$ (note that we define an artificial λ_{m+1} to be equal to 0). In this case the problem to be solved is

$$\varrho_\lambda \coloneqq \min_{x \in \mathbf{K}} \mathrm{MOM}_\lambda(x) \coloneqq \sum_{j=1}^{m} \lambda_j f_{(j)}(x).$$

We observe that for a fixed $x \in \mathbf{K}$, we can write the objective function as:

$$\mathrm{MOM}_\lambda(x) = \sum_{j=1}^{m} (\lambda_j - \lambda_{j+1}) S_j(x).$$

Then, we introduce the problem

$$\begin{aligned} \hat{\varrho_\lambda} \coloneqq \min \quad & \sum_{k=1}^{m} (\lambda_k - \lambda_{k+1}) S_k(x) \\ & t_k + r_{kj} \geq f_j(x), \quad j,k = 1, \ldots, m, \\ & r_{kj} \geq 0, \quad j,k = 1, \ldots, m, \\ & x \in \mathbf{K}. \end{aligned} \tag{11}$$

Let us denote

$$\overline{\mathbf{K}}_2 \coloneqq \{(x,t,r) \in \mathbb{R}^{d+m+m^2} :\ t_k + r_{kj} \geq f_j(x),\ r_{kj} \geq 0,\ r_{kj}^2 \leq (UB_j - LB)^2,\\ t_k^2 \leq \max\{UB^2, LB^2\},\ k,j = 1, \ldots, m,\ x \in \mathbf{K}\},$$

the basic closed semi-algebraic set that defines the feasible region of the Problem (11). (Note that similarly to the case of $\overline{\mathbf{K}}_1$ the augmented constraints in $\overline{\mathbf{K}}_2$ are also redundant.) Now, based in the previous lemma, it is straightforward to check the following result.

Lemma 8. *If* $\mathbf{K} \subset \mathbb{R}^d$ *satisfies Archimedean property then* $\overline{\mathbf{K}}_2 \subset \mathbb{R}^{d+m+m^2}$ *satisfies Archimedean property. Moreover* $\varrho_\lambda = \hat{\varrho_\lambda} = \min \{\sum_{k=1}^m (\lambda_k - \lambda_{k+1}) S_k(x) : (x,t,r) \in \overline{\mathbf{K}}_2\}$.

Another class of problems that can also be analyzed giving rise to a more compact formulation that the one in the general approach ($\mathbf{OMRP}_\lambda$) is the trimmed-mean problem. A trimmed-mean objective appears for $\lambda = (\overbrace{0, \ldots, 0}^{k_1}, 1, \ldots, 1, \overbrace{0, \ldots, 0}^{k_2})$.

This family of problems has attracted a lot of attention in last years in the field of location analysis because of its connections to robust solution concepts. Its rationale rests on the trimmed-mean concepts in statistics where the extreme observations (*outliers*) are removed to compute the central estimates (*mean*) of a sample. Thus, we are looking for a point x^* that minimizes the sum of the central functions, once we have excluded the k_2-smallest and the k_1-largest. Formally, the problem is

$$\varrho_{(k_1,k_2)} = \min_{x \in \mathbf{K} \subset \mathbb{R}^d} \sum_{i=k_1+1}^{m-k_2} f_{(i)}(x).$$

Now, we observe that $\sum_{i=k_1+1}^{m-k_2} f_{(i)}(x) = S_{m-k_2}(x) - S_{k_1}(x)$ (see (10) for the definition of S_k). Therefore, using the above transformation we have

$$S_{k_1}(x)=\max\left\{\sum_{j=1}^m \nu_j f_j(x):\sum_{j=1}^m \nu_j=k_1, 0\le \nu_j\le 1,\ \forall j\right\},$$

$$S_{n-k_2}(x)=\min\left\{(m-k_2)t+\sum_{j=1}^m r_j: t+r_j\ge f_j(x),\ r_j\ge 0,\ \forall j\right\}.$$

Thus, using both reformulations the trimmed-mean problem results in

$$\begin{aligned}\hat{\varrho}_{(k_1,k_2)} := \min \quad & (m-k_2)t+\sum_{j=1}^m r_j-\sum_{j=1}^m \nu_j f_j(x)\\ \text{s.t.} \quad & \sum_{j=1}^m \nu_j=k_1,\\ & t+r_j\ge f_j(x),\quad j=1,\ldots,m,\\ & r_j\ge 0,\quad j=1,\ldots,m,\\ & \nu_j(\nu_j-1)=0,\quad j=1,\ldots,m,\quad x\in\mathbf{K}.\end{aligned} \tag{(k1, k2)Tr}$$

Let us denote

$$\overline{\mathbf{K}}_3 := \Big\{(x,t,r,\nu)\in\mathbb{R}^{d+1+2m}:\ x\in\mathbf{K},\ t+r_j\ge f_j(x),\ r_j\ge 0,\ \nu_j^2-\nu_j=0,$$
$$r_j^2\le (UB_j-LB)^2,\ j=1,\ldots,m,$$
$$t^2\le \max\{UB^2,LB^2\},\ \sum_{j=1}^m \nu_j^2=k_1\Big\},$$

the basic closed semi-algebraic set that defines the feasible region of ((k_1,k_2)Tr).

Lemma 9. *If $\mathbf{K}\subset\mathbb{R}^d$ satisfies Archimedean property then $\overline{\mathbf{K}}_3\subset\ \mathbb{R}^{d+1+2m}$ satisfies Archimedean property. Moreover $\varrho_{(k_1,k_2)}=\hat{\varrho}_{(k_1,k_2)}=\min\{(m-k_2)t+\sum_{j=1}^m r_j-\sum_{j=1}^m \nu_j f_j(x):(x,t,r,\nu)\in\overline{\mathbf{K}}_3\}$.*

Remark 10. We point out that the above formulation also covers many others known problems. Among them, we mention the so-called "*Expropriation*" and "*Anti k-centrum problems*" [2,6]. The first problem consists on maximizing the value of the function that goes in a given sorted position, say k_2. Note that this objective is obtained in the general formulation allocating a lambda -1 to the position k_2 and zero everywhere else. Indeed, the objective function $-f_{(k_2)}(x)=S_{k_2}(x)-S_{k_2+1}(x)$. Therefore, the problem is

$$\begin{aligned}\hat{\varrho}_{(k_2,k_2+1)} := \min \quad & k_2 t+\sum_{j=1}^m r_j-\sum_{j=1}^m \nu_j f_j(x)\\ s.t. \quad & \sum_{j=1}^m \nu_j=k_2+1,\\ & t+r_j\ge f_j(x),\quad j=1,\ldots,m,\\ & r_j\ge 0,\quad j=1,\ldots,m,\\ & \nu_j(\nu_j-1)=0,\quad j=1,\ldots,m,\quad x\in\mathbf{K}.\end{aligned} \tag{Expro}$$

The second problem consists on maximizing the $(m-k)$-smallest values of the functions, i.e. $\max_{x\in K}\sum_{j=k}^m f_{(j)}(x)$. This problem is equivalent to $-\min_{x\in K}\sum_{j=k}^m -f_{(j)}(x)=-\min_{x\in K}(\sum_{j=1}^{k-1} f_{(j)}(x)-\sum_{j=1}^m f_j(x))$. Clearly, this last rewriting can be embedded into the formulation of ($(k-1,m)$Tr), even without the use of the ν variables!.

Remark 11. We observe that the special formulations for k-centrum (kC) and trimmed-mean ((k_1,k_2)Tr) are specially suitable for handling these two classes of problems. First of all, we note that if ((k_1,k_2)Tr) has $k_1=0$ the problem reduces to a k_2-centrum, variables ν_j are not needed and formulation ((k_1,k_2)Tr) simplifies exactly to (kC). Second, we point out that both formulations take advantage of the special structure of the considered problems and thus they are simpler than the general formulation ($\mathbf{OMRP}_\lambda$) applied to these problems. Actually, the number of variables in (kC), for solving the k-centrum problem (resp. ((k_1,k_2)Tr) for solving the trimmed-mean problem), is $d+1+m$ (resp. $d+2m+1$) while the number of variables for the same problem using ($\mathbf{OMRP}_\lambda$) is $d+m^2$. This reduction is remarkable due to the current status of nonlinear programming solvers. In spite of that, those problems, where no special structure is known or it cannot be exploited, can also be tackled using the general formulation ($\mathbf{OMRP}_\lambda$) at the price of using a larger number of variables.

3.2. *A convergence result for Problem* $\mathbf{OMRP}_\lambda^0$

We are now in position to develop a methodology to solve $\mathbf{OMRP}_\lambda^0$. Our approach defines a hierarchy of semidefinite relaxations based on Problem (7) whose solutions converge to the optimal solution of $\mathbf{OMRP}_\lambda^0$. Let $\mathbf{y}=(y_\alpha)$ be a real sequence indexed in the monomial basis $(x^\beta w^\gamma)$ of $\mathbb{R}[x,w]$ (with $\alpha=(\beta,\gamma)\in\mathbb{N}^d\times\mathbb{N}^{m^2}$). Let $p_\lambda(x,w)$ and $q_\lambda(x,w)$ be defined as in (8).

Let $h_0(x,w):=p_\lambda(x,w)$, and denote $\xi_j:=\lceil(\deg g_j)/2\rceil$, $j=1,\ldots,\ell$, $\nu_j:=\lceil(\deg h_j)/2\rceil$, $j=0,\ldots,m-1$ and $\nu'_j:=\lceil(\deg h'_j)/2\rceil=1$, $j=1,\ldots,m$; where $\{g_1,\ldots,g_\ell\}$ are the polynomial constraints that define $\mathbf{K}$ and $\{h_1,\ldots,h_{m-1}\}$ and $\{h'_1,\ldots,h'_m\}$ are, respectively, the polynomial constraints (3) and (4) in ($\mathbf{OMRP}_\lambda$).

Let us denote by $I(0)=\{1,\ldots,d\}$ and $I(j)=\{(j,k)\}_{k=1,\ldots,m}$, for all $j=1,\ldots,m$. With $x(I(0))$, $w(I(j))$ we refer, respectively, to the monomials x, w indexed only by subsets of elements in the sets $I(0)$ and $I(j)$, respectively. Then, for g_k, with $k=1,\ldots,\ell$, let $M_r(y;I(0))$ (respectively $M_r(g_k y;I(0))$) be the moment (resp. localizing) submatrix obtained from $M_r(y)$ (resp. $M_r(g_k y)$) retaining only those rows and columns indexed in the canonical basis of $\mathbb{R}[x(I(0))]$ (resp. $\mathbb{R}[x(I(0))]$). Analogously, for h_j, $j=1,\ldots,m-1$ and h'_j, $j=1,\ldots,m$, as defined in (3) and (4), respectively, let $M_r(y;I(0)\cup I(j)\cup I(j+1))$ (respectively $M_r(h_j y;I(0)\cup I(j)\cup I(j+1))$, $M_r(h'_j y;I(0)\cup I(j)\cup I(j+1))$) be the moment (resp. localizing) submatrix obtained from $M_r(y)$ (resp. $M_r(h_j y)$, $M_r(h'_j y)$) retaining only those rows and columns indexed in the canonical basis of $\mathbb{R}[x(I(0))\cup w(I(j))\cup w(I(j+1))]$ (resp. $\mathbb{R}[x(I(0))\cup w(I(j))\cup w(I(j+1))]$).

For $r\ge\max\{r_0,\nu_0\}$ where $r_0:=\max_{k=1,\ldots,\ell}\xi_k$ and $\nu_0:=\max\Big\{\max_{j=0,\ldots,m}\nu_j,\overbrace{\max_{j=1,\ldots,m}\nu'_j}^{=1}\Big\}=\max_{j=0,\ldots,m}\nu_j$, we introduce the following hierarchy of semidefinite programs:

$$\begin{aligned}\min_{\mathbf{y}} \quad & L_{\mathbf{y}}(p_\lambda)\\ \text{s.t.} \quad & M_r(\mathbf{y};I(0))\succcurlyeq 0,\\ & M_{r-\xi_k}(g_k\mathbf{y};I(0))\succcurlyeq 0,\quad k=1,\ldots,\ell,\\ & M_r(\mathbf{y};I(0)\cup I(j)\cup I(j+1))\succcurlyeq 0,\quad j=1,\ldots,m,\\ & M_{r-\nu_j}(h_j\mathbf{y};I(0)\cup I(j)\cup I(j+1))\succcurlyeq 0,\quad j=1,\ldots,m-1,\\ & M_{r-1}(h'_j\mathbf{y};I(0)\cup I(j)\cup I(j+1))\succcurlyeq 0,\quad j=1,\ldots,m,\\ & L_y\left(\sum_{i=1}^m w_{ij}-1\right)=0,\quad j=1,\ldots,m,\\ & L_y\left(\sum_{j=1}^m w_{ij}-1\right)=0,\quad i=1,\ldots,m,\\ & L_y(w_{ij}^2-w_{ij})=0,\quad i,j=1,\ldots,m,\\ & L_y(q_\lambda)=1,\end{aligned} \tag{Qr}$$

with optimal value denoted $\min\mathbf{Q}_r$.

Theorem 12. *Let $\overline{\mathbf{K}} \subset \mathbb{R}^{d+m^2}$ (as defined in (6)) be the feasible domain of ($\mathbf{OMRP}_\lambda$). Then, with the notation above*:

(a) $\min \mathbf{Q}_r \uparrow \rho_\lambda$ *as* $r \to \infty$.
(b) *Let* $\mathbf{y}^r$, *be an optimal solution of the SDP relaxation* ($\mathbf{Q}_r$). *Let* $\tilde{I}(j) = I(0) \cup I(j) \cup I(j+1)$ *for all* $j = 1, \ldots, m-1$. *If*

$$\operatorname{rank} M_r(\mathbf{y}^r; I(0)) = \operatorname{rank} M_{r-r_0}(\mathbf{y}^r; I(0))$$
$$\operatorname{rank} M_r(\mathbf{y}^r; \tilde{I}(j)) = \operatorname{rank} M_{r-v_0}(\mathbf{y}^r; \tilde{I}(j)), \quad j = 1, \ldots, m-1 \tag{12}$$

and if $\operatorname{rank}(M_r(y^r; I(0) \cup (I(k) \cup I(k+1)) \cap (I(j) \cup I(j+1)))) = 1$ *for all* $j \neq k$ *then* $\min \mathbf{Q}_r = \rho_\lambda$.

Moreover, let $\Delta_j := \{(x^*(j), w^*(j))\}$ *be the set of solutions obtained by the application of the condition (12). Then, every* (x^*, w^*) *such that* $(x_i^*, w_i^*)_{i \in I(j)} = (x^*(j), w^*(j))$ *for some* Δ_j *is an optimal solution of Problem* $\mathbf{OMRP}_\lambda$.

Proof. The convergence of the semidefinite relaxation ($\mathbf{Q}_r$) was proved by Jibetean and De Klerk [13, Theorem 9] for a general rational function over a basic, closed semi-algebraic set that satisfies Archimedean Property. Here, we use that result applied to the rational function in (9). Moreover, the index set of the indeterminates in the feasible set that generate localizing constraints, namely constraints (3) and (4) admits the decomposition $I(k)$, $k = 0 \ldots, m$ that satisfies the running intersection property (see [20, (1.3)]) and therefore, the result follows by combining [20, Theorem 3.2] and the results in [13]. □

The above theorem allows us to approximate and solve the original problem $\mathbf{OMRP}_\lambda^0$ by its relaxation ($\mathbf{Q}_r$) up to any degree of accuracy by solving block diagonal (sparse) SDP programs which are convex programs for each fixed relaxation order r and that can be solved, up to any given accuracy, in polynomial time with available open source solvers as SeDuMi, SDPA, SDPT3 [16], etc.

4. Generalized location problems with rational objective functions

This section considers a wide family of continuous location problems that has attracted a lot of attention in the recent literature of location analysis but for which there is not a common solution approach. The challenge is to design a common solution method to solve them for different distances in finite dimensional spaces.

We are given a set $A = \{a_1, \ldots, a_n\} \subset \mathbb{R}^d$ endowed with an ℓ_τ-norm (here ℓ_τ stands for the norm $\|x\|_\tau = (\sum_{i=1}^d |x_i|^\tau)^{1/\tau}$, for all $x \in \mathbb{R}^d$); and a feasible domain $\mathbf{K} := \{x \in \mathbb{R}^d : g_j(x) \geq 0, j = 1, \ldots, \ell\} \subset \mathbb{R}^d$, assumed to be a closed semi-algebraic set. Since we are interested in solving location problems we shall assume without loss of generality that we wish to solve the problem in a bounded domain so that $\mathbf{K}$ is compact. The goal is to find a point $x^* \in \mathbf{K}$ minimizing some globalizing function of the distances to the set A. Here, we consider that the globalizing function is rather general and that it is given as a rational function.

Some well-known examples, that are formulated in the above terms, are listed below (see e.g. [1,4,10,24] or [28]):

- $f(u_1, \ldots, u_n) = \sum_{i<j}^n |u_i - u_j|$, absolute deviation or envy problem.
- $f(u_1, \ldots, u_n) = \sum_{i=1}^n (u_i - \overline{u})^2$, variance problem.
- $f(u_1, \ldots, u_n) = \sum_{j=1}^n (w_j / u_j^2)$, obnoxious facility location.
- $f(u_1, \ldots, u_n) = \sum_{j=1}^n b_j / (1 + h_j |u_j|^\lambda)$, Huff competitive location.

The main feature and what distinguishes location problems from other general purpose optimization problems is that the dependence of the decision variables is given throughout the norms to the demand points in A, i.e. $\|x - a_i\|_\tau$. In this section, we consider a generalized version of continuous single facility location problems with rational objective functions over closed semi-algebraic feasible sets.

Let $f_j(u) := p_j(u)/q_j(u) : \mathbb{R}^n \mapsto \mathbb{R}$, with $p_j(u), q_j(u) \in \mathbb{R}[u_1, \ldots, u_n]$, $q_j(u) > 0$ for all $j = 1, \ldots, m$. We shall define the dependence of f_j to the decision variable $x \in \mathbb{R}^d$ via $u = (u_1, \ldots, u_n)$, where $u_i : \mathbb{R}^d \mapsto \mathbb{R}$, $u_i(x) := \|x - a_i\|_\tau$, $i = 1, \ldots, n$. Therefore, the j-th component of the ordered median objective function of our problems reads as

$$\tilde{f}_j(x) : \mathbb{R}^d \mapsto \mathbb{R}$$

$$x \mapsto \tilde{f}_j(x) := f_j(\|x - a_1\|_\tau, \ldots, \|x - a_n\|_\tau).$$

Consider the following problem:

$$\rho_\lambda := \min_x \left\{ \sum_{j=1}^m \lambda_j(x) \tilde{f}_{(j)}(x) : x \in \mathbf{K} \right\}, \tag{LOCOMRF}$$

where:

- $\lambda_j(x) \in \mathbb{R}[x]$, $j = 1, \ldots, m$, are generic polynomials.
- $\mathbf{K} := \{x \in \mathbb{R}^d : g_j(x) \geq 0, j = 1, \ldots, \ell\} \subseteq \mathbb{R}^d$ satisfies Archimedean property,
- $\tau := r/s$, $r, s \in \mathbb{N}$, $r \geq s$ and $gcd(r,s) = 1$.

First of all, since $\mathbf{K}$ is compact there exist $M' > 0$ such that $\|x\|_2 \leq M'$ for all $x \in \mathbf{K}$. Then, we observe that any feasible solution of (**LOCOMRF**) satisfies $\|x - a_i\|_2 \leq M' + \|a_i\|_2 \leq M' + \max_{1 \leq i \leq n} \|a_i\|_2 := M$. Then, since all norms are equivalent in $\mathbb{R}^d$, there exists $\gamma > 0$ such that $\|x\|_{2\tau} / \|x\|_2 \leq \gamma$, for all $x \in \mathbb{R}^d$. Hence, $\|x - a_i\|_{2\tau} \leq \gamma M := \overline{M}$. This bound will allow us to derive the constraints (17) of our reformulation of Problem (**LOCOMRF**). These constraints ensure that the feasible region is bounded which in our framework is sufficient to imply compactness. For this reason, we will call them from now on *compactness* constraints.

Next, Problem (**LOCOMRF**) does not reduce to the family $\mathbf{OMRP}_\lambda$ considered in Section 3 since the dependence on the decision variable x is not given in the form of polynomials. Note that ℓ_τ-norms are not, in general, polynomials.

To avoid this inconvenience, we introduce the following auxiliary problem:

$$\overline{\rho}_\lambda = \min_{x,w,u,v} \quad \sum_{j=1}^m \lambda_j(x) \sum_{i=1}^m f_i(u) w_{ij} \tag{13}$$

$$\text{s.t.} \quad \sum_{j=1}^m w_{ij} = 1 \quad \text{for } i = 1, \ldots, m,$$
$$\sum_{i=1}^m w_{ij} = 1 \quad \text{for } j = 1, \ldots, m,$$
$$\sum_{i=1}^m w_{ij} f_i(u) \geq \sum_{i=1}^m w_{ij+1} f_i(u), \quad j = 1, \ldots, m-1,$$
$$w_{ij}^2 - w_{ij} = 0 \quad \text{for } i,j = 1, \ldots, m,$$
$$v_{kl}^s \geq (x_k - a_{kl})^r, \quad k = 1, \ldots, n, \ l = 1, \ldots, d, \tag{14}$$

$$v_{kl}^s \geq (a_{kl} - x_l)^r, \quad k = 1, \ldots, n, \ l = 1, \ldots, d, \tag{15}$$

$$u_k^r = \left(\sum_{l=1}^d v_{kl} \right)^s, \quad k = 1, \ldots, n,$$
$$\sum_{j=1}^m w_{ij}^2 \leq 1, \quad i = 1, \ldots, n, \tag{16}$$

$$\sum_{j=1}^{m} v_{ij}^2 \le \overline{M}^{2\tau}, \quad i=1,\ldots,n, \tag{17}$$

$$w_{ij} \in \mathbb{R}, \quad \forall\, i,j=1,\ldots,m,$$
$$v_{kl} \in \mathbb{R}, \quad u_k \in \mathbb{R}, \quad k=1,\ldots,n,\ l=1,\ldots,d,$$
$$x \in \mathbf{K}.$$

We note in passing that constraints (16) and (17) are redundant but it is convenient to add them to ensure the Archimedean condition for the feasible region $\overline{\mathbf{K}}$ of the above auxiliary problem.

We also observe that the above problem simplifies for those cases where r is even. In these cases, we can replace the two sets of constraints, namely (14) and (15) by the simplest constraint

$$v_{kl}^s = (x_k - a_{kl})^r, \quad \forall\, k,l.$$

This reformulation reduces by $(n \times d)$ the number of constraints defining the feasible set. Moreover, these constraints do not induce semidefinite constraints in the moment approach but linear equations for substitution. Following the same scheme of the proof in Theorem 4 we get the following result that shows the equivalence between the above polynomial optimization formulation and our location problem (13).

Theorem 13. *Let x be a feasible solution of* (**LOCOMRF**) *then there exists a solution (x,u,v,w) for (13) such that their objective values are equal. Conversely, if (x,u,v,w) is a feasible solution for (13) then there exists a solution (x) for* (**LOCOMRF**) *having the same objective value. In particular $\rho_\lambda = \overline{\rho}_\lambda$. Moreover, if $\mathbf{K} \subset \mathbb{R}^d$ satisfies Archimedean property then $\overline{\mathbf{K}} \subset \mathbb{R}^{d+m^2+n(d+1)}$ also satisfies Archimedean property.*

Now, we can prove a convergence result that allows us to solve, up to any degree of accuracy, the above class of problems. Let $\mathbf{y} = (y_\alpha)$ be a real sequence indexed in the monomial basis $(x^\beta u^\gamma v^\delta w^\zeta)$ of $\mathbb{R}[x,u,v,w]$ (with $\alpha = (\beta,\gamma,\delta,\zeta) \in \mathbb{N}^d \times \mathbb{N}^n \times \mathbb{N}^{nd} \times \mathbb{N}^{m^2}$).

Let $h_0(x,u,v,w) := p_\lambda(x,u,v,w)$, and denote $\xi_j := \lceil(\deg g_j)/2\rceil$ and $\nu_j := \lceil(\deg h_j)/2\rceil$, where $\{g_1,\ldots,g_\ell\}$, and $\{h_1,\ldots,h_{3m-1+m^2+n(2d+3)}\}$ are, respectively, the polynomial constraints that define $\mathbf{K}$ and $\overline{\mathbf{K}}\backslash\mathbf{K}$ in (13). For $r \ge r_0 := \max\{\max_{k=1,\ldots,\ell}\xi_k, \max_{j=0,\ldots,3m-1+m^2+n(2d+3)}\nu_j\}$, we introduce the hierarchy of semidefinite programs:

$$\begin{array}{lll} \min\limits_{\mathbf{y}} & L_{\mathbf{y}}(p_\lambda) & \\ \text{s.t.} & M_r(\mathbf{y}) \succcurlyeq 0, & \\ & M_{r-\xi_k}(g_k,\mathbf{y}) \succcurlyeq 0, \quad k=1,\ldots,\ell, & (\overline{\mathbf{Q}}_r) \\ & M_{r-\nu_j}(h_j,\mathbf{y}) \succcurlyeq 0, \quad j=1,\ldots,3\,m-1+m^2+n(2d+3), & \\ & L_y(q_\lambda)=1, & \end{array}$$

with optimal value denoted $\min\overline{\mathbf{Q}}_r$.

Theorem 14. *Let $\overline{\mathbf{K}} \subset \mathbb{R}^{d+m^2+n(d+1)}$ be the feasible domain of Problem (13). Then, with the notation above:*

(a) $\min \overline{\mathbf{Q}}_r \uparrow \rho_\lambda$ *as* $r \to \infty$.
(b) *Let $\mathbf{y}^r$ be an optimal solution of the SDP relaxation $(\overline{\mathbf{Q}}_r)$. If*

$$\operatorname{rank} M_r(\mathbf{y}^r) = \operatorname{rank} M_{r-r_0}(\mathbf{y}^r) = t$$

then $\min \overline{\mathbf{Q}}_r = \rho_\lambda$ and one may extract t points $(x^*(k),u^*(k),v^*(k),w^*(k))_{k=1}^t \subset \overline{\mathbf{K}}$, all global minimizers of Problem (**LOCOMRF**).

Proof. The convergence of the semidefinite relaxation $(\overline{\mathbf{Q}}_r)$ follows from a result by Jibetean and De Klerk [13, Theorem 9] that it is applied here to the rational function in (13) and the closed *semi-algebraic* set $\overline{\mathbf{K}}$. The second assertion on the rank condition, for extracting optimal solutions, follows from applying [23, Theorem 5.7] to the SDP relaxation $(\overline{\mathbf{Q}}_r)$. □

Here, we also observe that one can exploit the block diagonal structure of the problem since there is a sparsity pattern in the variables of formulation (13). The reader may note that the only monomials that appear in that formulation are of the form $x^\alpha u_i^\beta \prod_{j=1}^m v_{ij}^{\gamma_j}$ for all $i=1,\ldots,m$. Hence, a result similar to Theorem 12 also holds for the hierarchy (Q_r) of SDP applied to the location problem. Nevertheless, although we have used it in our computational test, we do not give specific details for the sake of presentation and because of the similarity with Theorem 12.

We illustrate the above results with an instance of the well-known Weber problem with ℓ_3-norm and for 20 random demand points in $\mathbb{R}^3$.

Example 15. Let $A = \{(0.0758,0.0540,0.5308), (0.7792, 0.9340, 0.1299), (0.5688,0.4694,0.0119), (0.3371,0.1622,0.7943), (0.3112, 0.5285,0.1656), (0.6020,0.2630,0.6541), (0.6892,0.7482,0.4505), (0.0838,0.2290,0.9133), (0.1524,0.8259,0.5383), (0.9961,0.0782, 0.4427), (0.1066,0.9619,0.0046), (0.7749,0.8173,0.8687), (0.0844, 0.3998, 0.2599), (0.8000,0.4314,0.9106), (0.1818,0.2638,0.1455), (0.1361,0.8693,0.5797), (0.5499,0.1450,0.8530), (0.5499,0.1450, 0.8530), (0.4018,0.0760,0.2399), (0.1233,0.1839,0.2400)\}$ be a set of random points in $\mathbf{K} := [\mathbf{0},\mathbf{1}]^3$.

Then, we consider the problem

$$\begin{array}{ll} \min & \sum\limits_{a\in A} \|x-a\|_3 \\ \text{s.t.} & x \in \mathbf{K}. \end{array}$$

The exact optimal solution is given by $\overline{x} = (0.426397,0.438730, 0.455857)$ with optimal value $\overline{f} = 8.729976$. We get with our approach for the first relaxation of the problem, using SDPT3 [16], an optimal solution $x^* = (0.426397,0.438730,0.455857)$, with optimal value $f^* = 8.729976$. Thus, the relative error is $\overline{\epsilon} = |f^* - \overline{f}|/\overline{f} = 2.199595 \times 10^{-13}$.

For the same set of points, we consider a modification of the above problem by adding an extra *nonconvex* constraint:

$$\begin{array}{ll} \min & \sum\limits_{a\in A} \|x-a\|_3 \\ \text{s.t.} & x_1^2 - 2x_2^2 - 2x_3^2 \ge 0, \\ & x \in \mathbf{K}. \end{array}$$

The exact optimal solution of this problem is $\tilde{x} = (0.562304, 0.266296,0.295262)$ with optimal value $\tilde{f} = 10.109333$. The reader may note that the original solution $\overline{x}$ is not feasible for the new problem. Using our approach, again for the first relaxation order, we get $x^{**} = (0.562304,0.266296,0.295262)$ with optimal value $f^{**} = 10.109333$. Hence, the relative error in this case is $\tilde{\epsilon} = |f^{**} - \tilde{f}|/\tilde{f} = 5.801151 \times 10^{-9}$.

We show in Fig. 1 the feasible region of the later problem as well as the demand points and the optimal solutions (the exact and the ones obtained with our relaxed formulations) of the problems. The demand points in A are represented by '∗', the optimal solution, x^*, of the SDP relaxation without the non-convex constraint by '■' and the optimal solution, x^{**}, of the SDP relaxation with the non-convex constraint is depicted by '•'.

In the following, we recall the definition of some well-known location problems that will be the basis of our computational results. We observe that some of these problems admit different reformulations. Nevertheless, our goal is to test the efficiency of the general methodology when applied to some standard location problems. Improved results, specifically devoted to convex ordered median location problems under general ℓ_τ norms are

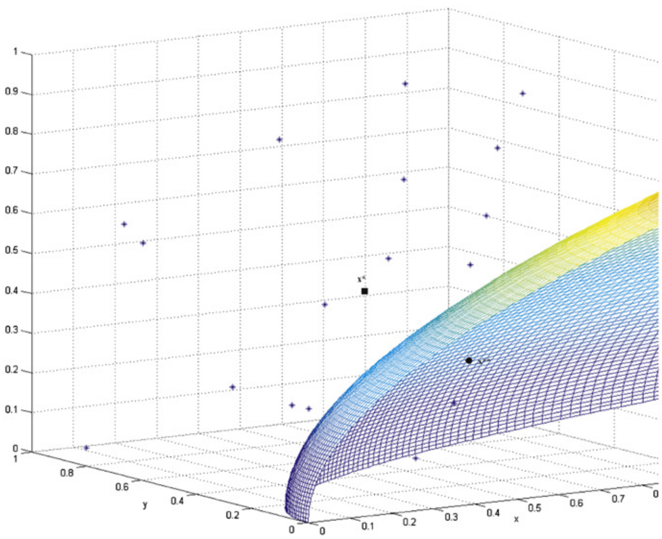

Fig. 1. Feasible region, demand points and optimal solutions of Example 15.

currently under research and will be the scope of a follow up paper.

We consider five problems types, namely Weber, center, k-centrum, range and trimmed-mean problems over a compact semi-algebraic set $\mathbf{K}$.

In the standard version of the *Weber problem*, we are given a set of demand points $\{a_1,\dots,a_n\}$ in $\mathbb{R}^d$ and a set of non-negative weights $\omega_1,\dots,\omega_n$ and one looks for a point $x^*\in\mathbf{K}\subseteq\mathbb{R}^d$ minimizing the weighted Euclidean distance from the demand point. In other words, the problem is: $\min_{x\in\mathbf{K}}\sum_{i=1}^n\omega_i\|x-a_i\|_\tau$. This problem has been largely studied in the literature of Location Analysis and perhaps its most well-known algorithm, in the unrestricted case, is the so-called Weiszfeld algorithm (see [44]). Here, we observe that this problem corresponds to a very particular choice of the elements in (**LOCOMRF**): $\lambda=(1,\dots,1)$, $f_i(u)=\omega_i u_i$.

The *minimax* (*center*) *location problem* looks for the location of a server $x\in\mathbf{K}$ that minimizes the maximum weighted distance to a given set of demands points $\{a_1,\dots,a_n\}$ in $\mathbb{R}^d$. Formally, the problem can be stated as: $\min_{x\in\mathbf{K}\subseteq\mathbb{R}^d}\max_{i=1,\dots,n}\omega_i\|x-a_i\|_\tau$, for some weights $\omega_1,\dots,\omega_n\geq 0$.

Once more, this problem has been extensively analyzed in the literature of Location Analysis and the most well-known algorithms to solve its unrestricted version are those by Elzinga–Hearn (only valid in $\mathbb{R}^2$ with Euclidean distance) and Dyer [9,8] and Megiddo [26] which are polynomial in fixed dimension. Again, we observe that this problem corresponds to a very particular choice of the elements in (**LOCOMRF**): $\lambda=(1,0,\dots,0)$, $f_i(u)=\omega_i u_i$.

The *k-centrum location* problem consists of finding the point x^* that minimizes the sum of the k-largest distances with respect to a given set of demands points $\{a_1,\dots,a_n\}$ in $\mathbb{R}^d$. Formally, the problem can be stated as: $\min_{x\in\mathbf{K}}\sum_{i=1}^k d_{(i)}(x)$, where $d_{(i)}(x)=\|x-a_{\sigma(i)}\|_\tau$ for a permutation σ such that $d_{\sigma(1)}(x)\geq\cdots\geq d_{\sigma(n)}(x)$. This problem has been considered in several papers and textbooks (see [28,5]). Currently, there exist few approaches to solve even the unrestricted version in the plane (i.e. $d=2$) and with the Euclidean norm (see [6,7,38]). The objective function of this problem is described by a vector of λ-parameters $\lambda=(\overbrace{1,\dots,1}^{k}$, $0,\dots,0)$, $f_i(u)=u_i$.

The next problem that we address in our computational experiments is the *range location problem*. This problem consists of minimizing the difference (range) between the maximum and minimum distances with respect to a given set of demand points $\{a_1,\dots,a_n\}$ in $\mathbb{R}^d$ (see [6,7,28]). Formally, the problem can be stated as: $\min_{x\in\mathbf{K}}[\max_{i=1,\dots,n}\|x-a_i\|_\tau-\min_{i=1,\dots,n}\|x-a_i\|_\tau]$. This problem corresponds to the following choice of the elements in (**LOCOMRF**): $\lambda=(1,0,\dots,0,-1)$, $f_i(u)=u_i$.

Finally, the (k_1,k_2)-*trimmed-mean location problem* looks for a point $x^*\in\mathbf{K}$ that minimizes the sum of the central distances, once we have excluded the k_2 closest and the k_1 furthest points. Formally, the problem is: $\min_{x\in\mathbf{K}}\sum_{i=k_1+1}^{n-k_2}d_{(i)}(x)$, where $d_{(i)}(x)=\|x-a_{\sigma(i)}\|_\tau$ for a permutation σ such that $d_{\sigma(1)}(x)\geq\cdots\geq d_{\sigma(n)}(x)$. This problem has been considered in several papers and textbooks (see [28,5]). Currently, there exists two approaches to solve it in the plane (i.e. $d=2$) and with the Euclidean norm (see [6,7]). The objective function of this problem, in terms of the elements in (**LOCOMRF**), is described by a vector of λ-parameters $\lambda=(\overbrace{0,\dots,0}^{k_1},1,\dots,1,\overbrace{0,\dots,0}^{k_2})$, $f_i(u)=u_i$.

We note in passing that, according to our discussion in Remark 10, the trimmed-mean problem type also covers the so-called *Expropriation and Anti k-centrum problems* [2,6] among others.

4.1. A finite convergence result for some location problems

In the following we will prove a specialized convergence result for a family of location problems. For this reason, we will introduce an alternative relaxation that is convenient for the considered location problems. Following the formulation in (kC), let us consider the following reformulation of the k-centrum location problem:

$$\begin{aligned}\hat{\varrho}:=\min\quad & kt+\sum_{i=1}^n r_i\\ \text{s.t.}\quad & z_i^2\geq\sum_{j=1}^d(x_j-a_{ij})^2,\quad i=1,\dots,n,\\ & t+r_i\geq z_i,\quad i=1,\dots,n,\\ & \sum_{j=1}^d x_j^2+z_i^2+t^2+r_i^2\leq M,\quad i=1,\dots,n, \qquad (18)\\ & t,r_i,z_i\geq 0,\quad i=1,\dots,n,\\ & x\in\mathbf{K}\subseteq\mathbb{R}^d.\end{aligned}\tag{kCP}$$

For $i=1,\dots,n$, let us denote by dis_i, the constraints $z_i^2-\sum_{j=1}^d(x_j-a_{ij})^2\geq 0$, by h_i, $i=1,\dots,n$ the compactness constraints (18), by lin_i, $i=1,\dots,n$ the linear inequality constraints and by $f=kt+\sum_{i=1}^n r_i$ the objective function, respectively, in Problem (**kCP**). In addition, let $\xi_j:=\lceil(\deg g_j)/2\rceil$ where $\{g_1,\dots,g_\ell\}$ are the polynomial constraints that define $\mathbf{K}$ in Problem (**kCP**). For $r\geq r_0:=\max\{\max_{k=1,\dots,\ell}\xi_k,1\}$, we introduce the following relaxation of Problem (**kCP**):

$$\begin{aligned}\varrho_r:=\min\quad & L_y(f)\\ \text{s.t.}\quad & M_r(y)\succcurlyeq 0,\quad i=1,\dots,n,\\ & M_{r-\xi_k}(g_k\mathbf{y})\succcurlyeq 0,\quad k=1,\dots,\ell,\\ & M_{r-1}(dis_i y)\succcurlyeq 0,\quad i=1,\dots,n,\\ & M_{r-1}(h_i y)\succcurlyeq 0,\quad i=1,\dots,n,\\ & L_y(lin_i)\geq 0,\quad i=1,\dots,n,\\ & L_y(x_j)=x_j,\quad j=1,\dots,d,\\ & L_y(r_i)=r_i,\quad i=1,\dots,n,\\ & L_y(z_i)=z_i,\quad i=1,\dots,n,\\ & L_y(t)=t. \qquad (19)\end{aligned}\tag{Relaxkr}$$

For $i=1,\dots,n$, we define the matrices

$$C_i=\begin{pmatrix} \sqrt{M}-x_1 & x_2 & \cdots & x_d & z_i & t & r_i \\ x_2 & \sqrt{M}+x_1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ x_d & 0 & \dots & \sqrt{M}+x_1 & 0 & 0 & 0 \\ z_i & 0 & \cdots & 0 & \sqrt{M}+x_1 & 0 & 0 \\ t & 0 & \cdots & 0 & 0 & \sqrt{M}+x_1 & 0 \\ r_i & 0 & \cdots & 0 & 0 & 0 & \sqrt{M}+x_1 \end{pmatrix}$$

and

$$D_i=\begin{pmatrix} z_i-(x_1-a_{i1}) & x_2-a_{i2} & \cdots & x_d-a_{id} \\ x_2-a_{i2} & z_i+(x_1-a_{i1}) & & 0 \\ \vdots & & \ddots & \\ x_d-a_{id} & 0 & & z_i+(x_1-a_{i1}) \end{pmatrix}.$$

Based on the above matrices, set $L_y(D_i) := (L_y(D_i(k,l)))_{k,l=1}^d$ and $L_y(C_i) := (L_y(C_i(k,l)))_{k,l=1}^{d+3}$.

Theorem 16. *Let y^* and ϱ_2 be an optimal solution and the optimal value of the first order feasible relaxation $(r=2)$ of the unrestricted versions ($\mathbf{K}=\mathbb{R}^d$) of Problem ($\mathbf{Relax_r^k}$) for Problem (**kCP**) reinforced with the constraints $L_y(C_i)\succcurlyeq 0$ and $L_y(D_i)\succcurlyeq 0$, for all $i=1,\dots,n$. Then, ϱ_2 is the optimal value and $x^* := (L_{y^*}(x_i))_{i=1}^d$ is an optimal solution of the corresponding Problem (**kCP**).*

Proof. First of all, since in the considered problem the lambda-vector is non-increasing monotone the unrestricted version of (**kCP**) is convex [12, Theorem 3.6]; and, by [28, Proposition 6.4] the optimal solution of this problem is attained at some point in $\mathbb{R}^d$. Consider the relaxation ($\mathbf{Relax_r^k}$) where all the constraints in the original problems are rewritten with moment variables.

Now, we note that all the constraints that were linear in the original problem are SDP representable and their representation in ($\mathbf{Relax_r^k}$) is exact. Next, we observe that the constraints $z_i^2 \geq \omega_i^2\|x-a_i\|_2^2 = \omega_i^2\sum_{j=1}^d (x_j-a_{ij})^2$, $i=1,\dots,n$ can be equivalently written, by the Schur complement, as $D_i\succcurlyeq 0$. Indeed, $D_i\succcurlyeq 0$ if and only if:

$$z_i-(x_1-a_{i1})-\frac{1}{z_i+(x_1-a_{i1})}(x_2-a_{i2},\dots,x_d-a_{id})\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}\begin{pmatrix}x_2-a_{i2}\\ \vdots \\ x_d-a_{id}\end{pmatrix}\geq 0$$

which can be rewritten as

$$z_i-(x_1-a_{i1})-(z_i+(x_1-a_{i1}))^{-1}\sum_{j=2}^d (x_j-a_{ij})^2 \geq 0.$$

This is clearly equivalent to the original constraint $z_i^2-\sum_{j=1}^d (x_j-a_{ij})^2 \geq 0$.

Analogously, the compactness constraints h_i of Problem (**kCP**) can be also written as $C_i\succcurlyeq 0$. Reinforcing the relaxation with $L_y(D_i)\succcurlyeq 0$ makes redundant the localizing constraints $M_r(d_i y)\succcurlyeq 0$ associated to the original constraints $z_i^2 \geq \omega_i^2\|x-a_i\|_2^2$ since $L_y(D_i)\succcurlyeq 0$ is an exact representation of that constraint. Analogously, reinforcing with $L_y(C_i)\succcurlyeq 0$ makes also redundant the localizing constraint $M_r(h_i y)\succcurlyeq 0$.

Then, we have that the reinforced version of ($\mathbf{Relax_r^k}$) is an exact SDP representation of Problem (kCP). Then, Problem ($\mathbf{Relax_r^k}$) can be solved to obtain the exact optimal objective value and an optimal solution of the original problems. Moreover, if y^* is an optimal solution then $x^*=(L_{y^*}(x_j))_{j=1}^d$, i.e. the values of the moment variables that correspond to the original x-variables are the optimal solutions of the original problem. □

First of all, we remark that the above result does not extend to Problem ((k$_1$,k$_2$)Tr) since its formulation contains some restrictions that cannot be exactly represented by SDP constraints. Nevertheless, including the new constraints makes its representation tighter and improves the convergence. Moreover, we would like to point out that the above result is mainly of theoretical interest and shows that the standard approach can recognize the SDP structure of some problems after some reinforcement with valid SDP inequalities. We also note that Theorem 16, although is stated for the unrestricted versions of the problems, can be extended to the restricted version provided that **K** is convex and SDP representable.

5. Computational experiments

A series of computational experiments have been performed in order to evaluate the behavior of the proposed methodology. Programs have been coded in MATLAB R2010b and executed in a PC with an Intel Core i7 processor at 2x 2.93 GHz and 8 GB of RAM. The semidefinite programs have been solved by calling SDPT3 4.0[16]. All our codes are available upon request.

We run the algorithm for several well-known continuous location problems: Weber, center, k-center, trimmed-mean and range. For each of them, we obtain the CPU times for computing solutions as well as the gap with respect to upper bounds obtained with the battery of functions in `optimset` of MATLAB or the implementation by [6,7], which only provide approximations on the exact solutions (optimality cannot be certified). The reader may note that we solve relaxed problems that give lower bounds. Therefore, the gap of our lower bounds is computed with respect to upper bounds which implies that actually may be even better than the one reported.

In order to compute the accuracy of an obtained solution, we use the following measure for the error (see [43]):

$$\epsilon_{\text{obj}} = \frac{|\text{the optimal value of the SDP}-\text{fopt}|}{\max\{1,\text{fopt}\}}, \tag{20}$$

where fopt is the approximated optimal value obtained with the functions in `optimset` or the implementation by [6,7].

We have organized our computational experiments in five different problems types that coincide with those described previously in Section 4. Our test problems are generated to be comparable with previous results of some algorithms in the plane [6,7] but, in addition, we also consider problems in $\mathbb{R}^3$ and a battery of non-convex constrained problems. Thus, we report on randomly generated points on the unit square and in the unit cube. Depending on the problem, we have been able to solve different problem sizes. In all problems, we could solve instances with at least 1000 points for planar and 3-dimensional problems and with an average accuracy higher than 10^{-5}. (We remark that for instance we could solve instances with many more points for Weber and center problems with high precisions.)

Our goal is to present the results organized per problem type, framework space ($\mathbb{R}^2$ or $\mathbb{R}^3$) and norm (ℓ_2 and ℓ_3). We report results only on the first relaxation order since its accuracy is rather good. Needless to say that raising the relaxation order one gains some extra precision (as expected) at the price of higher CPU times. In spite of that, the considered problems seem to be very well-approximated even with the first relaxation (as shown by our results).

The results in our tables are the average of 10 runs for each size and problem type. In all cases our tables are organized in the same way. Rows give the results for the different number of demand points considered in the problems. Column `n` stands for the number of points considered in the problem. Next, we present 6 blocks of two columns each. These blocks correspond to the different problem types, namely Weber, center, k-centrum ($k=\lceil 0.1\texttt{n}\rceil$ and $\lceil 0.5\texttt{n}\rceil$), range and trimmed-mean. Within each

Table 1
Computational results for different location problems in $\mathbb{R}^2$ with norm ℓ_2.

n	Weber		Center		k-Centrum, k=0.1n		k-Centrum, k=0.5n		Range		Trimmed-mean	
	ℓ_2		ℓ_2		ℓ_2		ℓ_2		ℓ_2		ℓ_2	
	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}
10	0.31	0.00000127	1.33	0.00000978	1.34	0.00001760	1.34	0.00000455	1.26	0.01184986	2.98	0.00018438
20	0.68	0.00000005	3.08	0.00001456	3.31	0.00000598	3.18	0.00000111	2.21	0.06784203	6.34	0.00018729
30	1.00	0.00000003	5.35	0.00046734	6.34	0.00000465	5.50	0.00000123	3.10	0.02626473	9.96	0.00013896
50	1.70	0.00000005	10.61	0.00001725	11.97	0.00000425	13.22	0.00000048	6.57	0.07291619	20.89	0.00015183
100	3.55	0.00000004	30.83	0.00000542	38.59	0.00000292	37.58	0.00000020	14.58	0.02572793	46.62	0.00015415
200	7.05	0.00000004	84.16	0.00001519	99.55	0.00000093	100.39	0.00000044	31.34	0.03714671	118.09	0.00014847
300	10.66	0.00000003	139.36	0.00000386	164.28	0.00000055	159.49	0.00000005	74.49	0.03314587	188.91	0.00014136
400	14.27	0.00000003	216.28	0.00000337	240.42	0.00000057	211.09	0.00000010	94.59	0.04756016	304.58	0.00014574
500	17.74	0.00000003	305.36	0.00000336	328.64	0.00000028	285.02	0.00000012	172.06	0.05599743	391.78	0.00014832
1000	39.82	0.00000002	736.25	0.00002836	753.93	0.00000010	666.20	0.00000003	323.17	0.03572262	903.89	0.00016247

Table 2
Computational results for different location problems in $\mathbb{R}^2$ with norm ℓ_3.

n	Weber		Center		k-Centrum, k=0.1n		k-Centrum, k=0.5n		Range		Trimmed-mean	
	ℓ_3		ℓ_3		ℓ_3		ℓ_3		ℓ_3		ℓ_3	
	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}
10	0.44	0.00000029	1.70	0.00000441	1.46	0.00000998	1.45	0.00000512	1.38	0.01019686	2.87	0.00026887
20	1.01	0.00000007	3.59	0.00001389	3.71	0.00001100	4.15	0.00000065	2.70	0.02628318	6.75	0.00017690
30	1.50	0.00000044	6.33	0.00001259	6.46	0.00000321	6.93	0.00000056	5.35	0.09088091	11.19	0.00019343
50	2.50	0.00000018	12.91	0.00000947	13.92	0.00000554	16.20	0.00000048	10.51	0.07220939	20.62	0.00021732
100	5.21	0.00000012	34.07	0.00000690	42.11	0.00000256	34.41	0.00000040	24.30	0.03754705	52.83	0.00017720
200	10.73	0.00000010	87.18	0.00000663	111.38	0.00000043	98.39	0.00000028	55.67	0.04069077	128.14	0.00018684
300	16.07	0.00000008	173.36	0.00001240	180.18	0.00000067	157.35	0.00000017	92.37	0.07366743	191.46	0.00016696
400	21.30	0.00000015	240.12	0.00001163	262.77	0.00000053	233.61	0.00000010	154.74	0.02080770	312.34	0.00020440
500	27.46	0.00000010	299.41	0.00000498	341.34	0.00000035	291.80	0.00000006	168.54	0.01652014	391.24	0.00019197
1000	58.32	0.00000008	864.93	0.00009096	811.07	0.00000014	729.30	0.00000003	982.29	0.03131304	1023.53	0.00017985

Table 3
Computational results for different location problems in $\mathbb{R}^3$ with norm ℓ_2.

n	Weber		Center		k-Centrum, k=0.1n		k-Centrum, k=0.5n		Range		Trimmed-mean	
	ℓ_2		ℓ_2		ℓ_2		ℓ_2		ℓ_2		ℓ_2	
	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}
10	0.51	0.00000005	2.42	0.00003587	2.52	0.00005446	2.56	0.00006255	1.48	0.04627554	5.26	0.00135759
20	0.88	0.00000004	6.19	0.00012431	6.20	0.00000923	5.99	0.00000072	3.37	0.04179864	11.50	0.01742414
30	1.36	0.00000003	10.21	0.00006793	11.53	0.00000569	10.26	0.00000249	5.70	0.05373857	18.89	0.00014033
50	2.19	0.00000002	21.13	0.00008918	21.87	0.00000362	20.34	0.00000071	10.66	0.04831570	39.05	0.00013365
100	4.79	0.00000005	48.51	0.00001014	60.90	0.00000221	52.52	0.00000011	25.01	0.07934761	98.52	0.00011729
200	9.69	0.00000004	154.65	0.00027283	165.58	0.00000201	137.63	0.00000017	61.03	0.06438157	268.73	0.00011069
300	14.66	0.00000003	314.56	0.00000531	275.11	0.00000082	225.88	0.00000018	97.15	0.08691918	471.14	0.00011019
400	20.38	0.00000003	409.41	0.00024259	388.04	0.00000058	344.88	0.00000014	149.10	0.07838252	682.42	0.00011466
500	25.32	0.00000005	566.29	0.00007134	551.81	0.00000171	452.85	0.00000004	217.73	0.05275364	954.48	0.00010885
1000	56.86	0.00000003	1494.76	0.00024533	1362.24	0.00000053	1149.90	0.00000003	629.69	0.07637927	3472.17	0.00011673

block CPU time is the average running time needed to solve each of the instances and ϵ_{obj} gives the error measure (see (20)).

We report problems with up to 1000 demands points randomly generated in the unit square and the unit cube. We move n between 10 and 1000 and 10 instances were generated for each value of n. The first relaxation of the problems was solved in all cases. For the k-centrum problem type we considered two different k values to test the difficulty of problems with respect to that parameter, $k = \lceil 0.1\mathtt{n} \rceil, \lceil 0.5\mathtt{n} \rceil$.

Tables 1 and 2 (respectively 3 and 4) show the averages CPU times and gaps obtained for problems on the plane (respectively in the 3-dimensional space) with norms ℓ_2 and ℓ_3. From our tables we conclude that Weber problem is the simplest one whereas the trimmed-mean problem is the hardest one, as expected. We observe that for small values of k, i.e. $k = \lceil 0.1\mathtt{n} \rceil$ the k-centrum is slightly harder than for values closer to n. The results for the range problem are similar to those of the k-centrum problems both in CPU time and accuracy. Finally, the trimmed-mean problems are the hardest problems among the five considered problem types. We are able to solve similar sizes with similar accuracies using the first order relaxation. However, CPU times are significantly higher than for the other problem types. These results show that this methodology can be efficiently applied to solve medium sized location problems. We remark that CPU times increase linearly with the number of

Table 4
Computational results for different location problems in $\mathbb{R}^3$ with norm ℓ_3.

n	Weber		Center		k-Centrum, $k=0.1$n		k-Centrum, $k=0.5$n		Range		Trimmed-mean	
	ℓ_3		ℓ_3		ℓ_3		ℓ_3		ℓ_3		ℓ_3	
	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}
10	0.69	0.00000010	2.66	0.00001359	2.71	0.00005560	2.87	0.00000449	2.22	0.10307486	5.71	0.00480821
20	1.35	0.00000011	6.65	0.00000975	6.86	0.00001997	6.69	0.00000579	4.14	0.08095350	12.28	0.00018630
30	2.02	0.00000012	10.53	0.00001685	11.57	0.00001989	11.51	0.00000032	6.52	0.01220608	19.59	0.00019285
50	3.57	0.00000013	21.95	0.00007675	21.92	0.00000427	22.07	0.00000048	11.68	0.01228767	38.27	0.00017595
100	7.32	0.00000012	57.85	0.00003824	64.18	0.00000212	53.87	0.00000131	28.79	0.08973747	92.26	0.00018263
200	14.64	0.00000011	139.71	0.00007898	173.98	0.00000099	136.63	0.00000022	63.18	0.06295116	244.31	0.00017962
300	22.45	0.00000008	250.37	0.00001856	290.92	0.00000063	245.90	0.00000007	114.54	0.08206731	423.24	0.00017924
400	29.75	0.00000009	409.45	0.00016182	446.72	0.00000028	368.77	0.00000011	233.32	0.03201793	621.35	0.00018478
500	37.22	0.00000010	600.27	0.00003066	578.19	0.00000042	449.46	0.00000014	253.54	0.08371244	848.15	0.00016222
1000	84.06	0.00000008	1606.89	0.00009110	1513.15	0.00000023	1280.10	0.00000007	985.88	0.07694671	3127.34	0.00016783

Table 5
Computational results for different location problems in $\mathbb{R}^3$ with norm ℓ_3 and the non-convex constraint of Example 15.

n	Weber		Center		k-Centrum, $k=0.5$n		Range		Trimmed-mean	
	ℓ_3		ℓ_3		ℓ_3		ℓ_3		ℓ_3	
	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}
10	1.36	0.09405959	2.60	0.02665885	3.71	0.06430646	3.33	0.01200950	6.77	0.01612428
20	2.62	0.02005929	6.81	0.02853398	9.64	0.06002251	5.88	0.01654221	19.51	0.06695578
30	4.43	0.01289829	12.04	0.02398289	14.94	0.04207472	9.27	0.01746477	33.10	0.05719759
50	7.02	0.04547366	23.43	0.04546409	30.81	0.03988616	16.83	0.01842803	70.97	0.02065027
100	16.93	0.00878935	63.92	0.03436590	69.45	0.03902827	38.43	0.01780251	175.74	0.02007698
200	41.74	0.02680226	159.26	0.04300610	167.13	0.03277066	130.96	0.01943710	521.29	0.01346846
300	97.21	0.00284410	284.28	0.02752200	250.99	0.03194087	149.15	0.04979041	961.07	0.01450505
400	116.66	0.00830339	379.43	0.03601218	362.18	0.02811320	222.61	0.03480614	1585.15	0.01388022
500	153.45	0.03299625	465.89	0.05757692	461.47	0.03286206	410.30	0.02320407	1945.49	0.01294002
1000	343.82	0.08902627	1721.78	0.05060629	1422.53	0.03233196	982.99	0.06977909	8383.62	0.01229571

points in all problem types. A linear regression between these times and the number of points gives a regression coefficient R-squared (coefficient of determination of the regression) greater than 0.98 for all the problems. Therefore, this shows a linear dependence, up to the tested sizes, between problem sizes and CPU times for solving the corresponding relaxations.

Special attention is required to explain the results for the range problem instances. There the gaps are slightly larger than for the rest of the problems, actually some of these gaps are negative. We have tested, indeed, that this behavior is due to the fact that we got solutions with better objective values than those obtained by `optimset` which was considered our reference for comparison. This observation proves the remarkable behavior of our approach.

Finally, in Table 5 we also report results on the same problem types in the 3-dimensional space with norm ℓ_3 and the additional non-convex constraint used in Example 15. As the reader can see, our approach is able to solve similar problems sizes, with good accuracy and CPU times. Once more, CPU times increase almost linearly with size for all problem types using that relaxation order. This shows that this method is also applicable to constrained (non-convex) problems with different norms and it is not limited to planar or low dimensional framework spaces.

6. Conclusions

We develop a unified tool for minimizing ordered weighted averaging of rational functions. This approach goes beyond a trivial adaptation of the general theory of moments since ordered weighted averages of rational functions are, in general, neither rational functions nor the supremum of rational functions so that current results cannot directly be applied to handle these problems. As an important application we cast a general class of continuous location problems within the minimization of OWA rational functions. We report computational results that show the powerfulness of this methodology to solve medium size continuous location problems.

This new approach solves a broad class of convex and non-convex continuous location problems that, up to date, were only partially solved in the specialized literature. We have tested this methodology with some medium size standard ordered median location problems in the plane and in the 3-dimensional space. Our goal was not to compete with previous algorithms since most of them are either designed for specific problems or only applicable for planar problems. However, in all cases we obtained reasonable CPU times and accurate results even with low relaxation orders. Our good results heavily rely on the fact that we have detected sparsity patterns in these problems reducing considerably the sizes of the SDP objects to be considered.

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